IDEALS WHOSE ASSOCIATED GRADED RINGS ARE ISOMORPHIC TO THE BASE RINGS

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Abstract. Let k be a field. We determine the ideals I in a finitely generated graded k-algebra A, whose associated graded rings $\bigoplus_{n\geq 0} I^n/I^{n+1}$ are isomorphic to A. Also we compute the graded local cohomologies of the Rees rings $A[It]$ and give the condition for $A[It]$ to be generalized Cohen-Macaulay under the condition that A is generalized Cohen-Macaulay.
MSC: 13A30, 13D45

Introduction

Let k be a field and $S = k[X_1, \ldots, X_n]$ be a polynomial ring over k. Let $J \subset S$ be a homogeneous ideal generated in degree ≥ 2 and consider the finitely generated standard graded k-algebra $A = S/J$. Let $I = (\bar{f}_1, \ldots, \bar{f}_m) \subset A$ be a homogeneous ideal and, for $f \in S$, we denote by $\bar{f}$ the image of f in A. In particular, we denote $\bar{X}_i$ by x_i. We consider the associated graded ring $G = \bigoplus_{\ell=0}^{\infty} I^\ell/I^{\ell+1}$ and the Rees ring $R = \bigoplus_{\ell=0}^{\infty} I^\ell t^\ell = A[It]$ with regard to $I \subset A$. We set $R_+ = \bigoplus_{\ell=1}^{\infty} I^\ell t^\ell$.

It is well known that $A \cong G$ if $I = \mathfrak{m} := (x_1, \ldots, x_n)$. In this paper, we determine precisely the ideals $I \subset A$ satisfying $A \cong G$. Such ideals turn out to be generated by certain linear forms (Theorem 2). Next we compute graded local cohomologies of Rees algebra with regard to such papameter ideals, which is an extension of the result in [3] for $I = \mathfrak{m}$ (Proposition 6). As an application, we give the condition for generalized Cohen-Macaulayness of the Rees algebra when the base ring is also generalized Cohen-Macaulay (Theorem 8).

1. The Main Theorem

Consider the following natural surjective homomorphism from the polynomial ring $k[\overline{X}, \overline{Y}] = k[X_1, \ldots, X_n, Y_1, \ldots, Y_m](= S[Y_1, \ldots, Y_m])$ in the variables $X_1, \ldots, X_n, Y_1, \ldots, Y_m$:

$$\begin{array}{lcrcl} \psi & : & k[X_1, \ldots, X_n, Y_1, \ldots, Y_m] & \longrightarrow & G \\ & & X_i & \longmapsto & [x_i] \in A/I \\ & & Y_i & \longmapsto & [\bar{f}_i] \in I/I^2 \end{array} \tag{1}$$

where $[a]$ denotes the equivalent class of $a \in A$ in A/I or I/I^2. For an element $g \in k[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$, we will denote by $\deg_{\overline{Y}}(g)$ the degree of g with regard to the variables $Y_1, \ldots, Y_m$. $\operatorname{Ker} \psi$ is computed as follows.

Lemma 1. *$\operatorname{Ker}\psi = J + (f_1, \dots, f_m) + L$, where L is an ideal generated by the following set:*

$$\left\{ g \in k[\overline{X}, \overline{Y}] \ : \ \begin{array}{l} g \notin J + (f_1, \dots, f_m), \\ g \text{ is homogeneous in } \overline{X} \text{ and } \overline{Y} \text{ respectively}, \\ g(X_1, \dots, X_n, f_1, \dots, f_m) \in (f_1, \dots, f_m)^{d+1} + J \\ \text{with } d = \deg_{\bar{Y}}(g) (> 0) \end{array} \right\}$$

Proof. Consider the natural surjection:

$$\begin{array}{rcccc} \varphi & : & A[Y_1, \dots, Y_m] & \longrightarrow & G \\ & & a \ (\in A) & \longmapsto & [a] \in A/I \\ & & Y_i & \longmapsto & [\bar{f}_i] \in I/I^2 \end{array} \tag{2}$$

where we know that $IA[Y_1, \dots, Y_m] \subset \operatorname{Ker}\varphi$. From this we obtain the following commutative diagram:

$$\begin{array}{ccc} k[X_1, \dots, X_n, Y_1, \dots, Y_m] & \xrightarrow{\psi} & G \\ {\scriptstyle \bmod JS[\overline{Y}]} \big\downarrow & & \big\uparrow {\scriptstyle \{Y_i \mapsto [\bar{f}_i]\}_{i=1}^m} \\ A[Y_1, \dots, Y_m] & \xrightarrow{\bmod IA[\overline{Y}]} & (A/I)[Y_1, \dots, Y_m] \\ & & \| \\ & & \dfrac{k[X_1, \dots, X_n, Y_1, \dots, Y_m]}{(J + (f_1, \dots, f_m))} \end{array}$$

and we know that $\psi(g(\overline{X}, \overline{Y})) = g(\overline{X}, f_1, \dots, f_m) + (f_1, \dots, f_m)^{d+1} + J$ if $d = \deg_{\overline{Y}}(g)$. Then we know that $\operatorname{Ker}\psi$ is generated by the set

$$\mathcal{L}' = \{ g \in k[\overline{X}, \overline{Y}] \ : \ g(\overline{X}, f_1, \dots, f_m) \in (f_1, \dots, f_m)^{d+1} + J \text{ with } d = \deg_{\bar{Y}} g \}.$$

We also know that $J + (f_1, \dots, f_m) \subset \operatorname{Ker}\psi$. Since the generators of $J + (f_1, \dots, f_m)$ do not contain $Y_1, \dots, Y_m$, these generators are contained in $\mathcal{L}'$. $\square$

Now we show our main theorem.

Theorem 2. *$A \cong G$ if and only if I is generated by $x_{i_1}, \dots, x_{i_m}$ such that we can choose a set $\{h_1, \dots, h_p\}$ of homogeneous generators of J such that, for each j, $h_j \in k[X_{i_1}, \dots, X_{i_m}]$ or $h_j \in k[X_{i_{m+1}}, \dots, X_{i_n}]$, where $\{i_1, \dots, i_m, i_{m+1}, \dots, i_n\} = \{1, \dots, n\}$.*

Proof. We first prove the only-if part. We have

$$G \cong k[X_1, \dots, X_n, Y_1, \dots, Y_m]/(J + (f_1, \dots, f_m) + L)$$

where L is the ideal as given in Lemma 1. Now J and $(f_1, \dots, f_m)$ give relations on the variables $X_1, \dots, X_n$ and $f_i \notin J$ for all $i = 1, \dots, m$. Thus in order for G to be isomorphic to $A = k[X_1, \dots, X_n]/J$, we must have

- (i) $f_1, \dots, f_m$ are linear forms in $X_1, \dots, X_n$, say $f_1 = X_1, \dots, f_m = X_m$ for simplicity, and
- (ii) L precisely gives the relations on $Y_1, \dots Y_m$, that, when Y_i are replaced by X_i $(i = 1, \dots, m)$, give the relations on $X_1, \dots, X_m$ given by J.

In fact, if we have a non-linear relation f_j, then L must contain relations such as $\{X_i - Y_i\}_i \cup \{Y_i\}_i$ to remove the extra relations on $X_1, \dots, X_n$ caused by f_j. But such relations cannot be in L. Thus we know (i), and from this we also know (ii).

Now by (i), we can assume without loss of generality that $I = (x_1, \ldots, x_m)$ for some $m \le n$, i.e., $f_i = X_i$, $i = 1, \ldots, m$. Then we have $G \cong k[X_1, \ldots, X_n, Y_1, \ldots, Y_m]/(J + (X_1, \ldots, X_m) + L)$ where L is generated by the set

$$\mathcal{L} := \left\{ g \in k[\overline{X}, \overline{Y}] \ : \ \begin{array}{l} g(\overline{X}, \overline{Y}) \text{ is homogeneous in } \overline{X} \text{ and } \overline{Y} \text{ respectively} \\ g \notin J + (X_1, \ldots, X_m),\ \deg_{\overline{Y}}(g) = d > 0, \\ g(\overline{X}, X_1, \ldots, X_m) \in (X_1, \ldots, X_m)^{d+1} + J \end{array} \right\}$$

For an element $g \in \mathcal{L}$, if $g(\overline{X}, X_1, \ldots, X_m) \in (X_1, \ldots, X_m)^{d+1}$ with $d = \deg_{\overline{Y}}(g)$ then $g \in (X_1, \ldots, X_m)$, which is a contradition since $g \notin (X_1, \ldots, X_m) + J$. This we can assume that $g(\overline{X}, X_1, \ldots, X_m) \in J$. Thus

$$\mathcal{L} = \left\{ g \in k[\overline{X}, \overline{Y}] \ : \ \begin{array}{l} g \text{ is homogeneous in } \overline{X} \text{ and } \overline{Y} \text{ respectively,} \\ g \notin J + (X_1, \ldots, X_m), g(\overline{X}, X_1, \ldots, X_m) \in J \end{array} \right\}$$

and

$$\begin{aligned} G &\cong \frac{k[\overline{X}, \overline{Y}]/(X_1, \ldots, X_m)}{(J + L + (X_1, \ldots, X_m))/(X_1, \ldots, X_m)} \\ &\cong \frac{k[Y_1, \ldots, Y_m, X_{m+1}, \ldots, X_n]}{(J + L + (X_1, \ldots, X_m))/(X_1, \ldots, X_m)}. \end{aligned}$$

Now by (ii), we must have (a) we can choose a set of generators $h_1, \ldots, h_p$ of J as follows: $h_i \in (X_1, \ldots, X_m)$ or h_i contains no monomial from $(X_1, \ldots, X_m)$ for all i, and (b) $L = \sigma(J \cap (X_1, \ldots, X_m))k[Y_1, \ldots, Y_m, X_{m+1}, \ldots, X_n]$ with $\sigma : k[X_1, \ldots, X_n] \to k[Y_1, \ldots, Y_m, X_{m+1}, \ldots, X_n]$ such that $\sigma(X_i) = Y_i$ for $1 \le i \le m$ and $\sigma(X_i) = X_i$ for $m + 1 \le i \le n$. But by the condition (a), $\mathcal{L}$ can be chosen as follows:

$$\mathcal{L} = \left\{ g \in k[\overline{Y}] \ : \ g(\overline{Y}) \text{ is homogeneous, } g(X_1, \ldots, X_m) \in J \ \right\}$$

and the condition (b) is satisfied. Consequently, we must have the following: we can choose a set of homogeneous generators $h_1, \ldots, h_p$ of J such that as follows: $h_i \in k[X_1, \ldots, X_m]$ or $h_i \in k[X_{m+1}, \ldots, X_n]$ for each i.

The proof of the if-part is carried out by tracing the above discussion conversely. □

Now we have the following well-known result as a corollary.

Corollary 3. *If $I = \mathfrak{m}$, then $A \cong G$.*

2. Local cohomologies of Rees algebras

We compute here the local cohomology of our Rees algebra R in the case of $A \cong G$. For a graded ring $R = \bigoplus_{n \ge 0} R_n$ such that R_0 is local with the maximal ideal $\mathfrak{m}$ and a graded R-module M, we denote by $H^q_{\mathcal{M}}(M)$ the qth graded local cohomology, where $\mathcal{M} = \mathfrak{m} \oplus \bigoplus_{n \ge 1} R_n$.

Now we cite two results.

Theorem 4 (Herzog-Popescu-Trung [3]). *Let $A = k[X_1, \ldots, X_n]/J$ be a residue class ring with regard to a homogeneous ideal J and let $\mathfrak{m} = (x_1, \ldots, x_n)$. Then we have*

$$H^i_{\mathfrak{m}}(A[\mathfrak{m}t])_a = \begin{cases} H^i_{\mathfrak{m}}(A)_a^{\oplus a+1} & a \ge 0 \\ 0 & a = -1 \\ H^{i-1}_{\mathfrak{m}}(A)_a^{\oplus -a-1} & a \le -2 \end{cases}$$

Theorem 5 (Goto-Watanabe [2]). *Let R, S be graded rings defined over k and $\mathfrak{m} = R_+$, $\mathfrak{n} = S_+$ be their H-maximal ideals. We put $T = R \otimes_k S$ and $\mathcal{M} = T_+$. If M (resp. N) is a graded R- (resp. S-) module, we have*

$$H^q_{\mathcal{M}}(M \otimes_k N) = \bigoplus_{i+j=q} H^i_{\mathfrak{m}}(M) \otimes_k H^j_{\mathfrak{n}}(N)$$

Now, according to Theorem 2, we can assume that $I = (x_{i_1}, \ldots, x_{i_m}) \subset A$ and $A = S/J$ with $J = (h_1, \ldots, h_r, h_{r+1}, \ldots, h_p)$ such that $h_1, \ldots, h_r \in k[X_{i_1}, \ldots, X_{i_m}]$ and $h_{r+1}, \ldots, h_p \in k[X_{i_{m+1}}, \ldots, X_{i_n}]$. We further set $A_1 = k[X_{i_1}, \ldots, X_{i_m}]/(h_1, \ldots, h_r)$, $\mathfrak{m}_1 = (x_{i_1}, \ldots, x_{i_m}) \subset A_1$ and $A_2 = k[X_{m+1}, \ldots, X_n]/(h_{r+1}, \ldots, h_p)$, $\mathfrak{m}_2 = (x_{i_{m+1}}, \ldots, x_{i_n}) \subset A_2$. Then we have $A = A_1 \otimes_k A_2$.

Proposition 6. *Let A and J be as above. Let $R = A[It]$ be the Rees algebra with regard to $I = (x_{i_1}, \ldots, x_{i_m}) \subset A$ where $X_{i_1}, \ldots, X_{i_m}$ are as in Theorem 2. Then we have*

$$\begin{aligned} H^\ell_{\mathcal{M}}(R)_a &= \bigoplus_{\substack{\alpha+\beta=a \\ \alpha \geq 0}} \bigoplus_{p+q=\ell} H^p_{\mathfrak{m}_1}(A_1)^{\oplus^{\alpha+1}}_{\alpha} \otimes_k H^q_{\mathfrak{m}_2}(A_2)_\beta \\ &\oplus \bigoplus_{\substack{\alpha+\beta=a \\ \alpha \leq -2}} \bigoplus_{p+q=\ell} H^{p-1}_{\mathfrak{m}_1}(A_1)^{\oplus^{-\alpha-1}}_{\alpha} \otimes_k H^q_{\mathfrak{m}_2}(A_2)_\beta \end{aligned}$$

where $\mathcal{M} = \mathfrak{m} \otimes R_+$.

Proof. We have $A[It] \cong A_1[\mathfrak{m}_1 t] \otimes_k A_2$. Thus by Theorem 5 and Theorem 4 we compute

$$\begin{aligned} H^\ell_{\mathcal{M}}(A[It])_a &= \bigoplus_{i+j=\ell} (H^i_{\mathfrak{m}_1}(A_1[\mathfrak{m}_1 t]) \otimes_k H^j_{\mathfrak{m}_2}(A_2))_a \\ &= \bigoplus_{i+j=\ell} \bigoplus_{\alpha+\beta=a} H^i_{\mathfrak{m}_1}(A_1[\mathfrak{m}_1 t])_\alpha \otimes_k H^j_{\mathfrak{m}_2}(A_2)_\beta \\ &= \bigoplus_{\substack{\alpha+\beta=a \\ \alpha \geq 0}} \bigoplus_{i+j=\ell} H^i_{\mathfrak{m}_1}(A_1)^{\oplus^{\alpha+1}}_{\alpha} \otimes_k H^j_{\mathfrak{m}_2}(A_2)_\beta \\ &\quad \oplus \bigoplus_{\substack{\alpha+\beta=a \\ \alpha \leq -2}} \bigoplus_{i+j=\ell} H^{i-1}_{\mathfrak{m}_1}(A_1)^{\oplus^{-\alpha-1}}_{\alpha} \otimes_k H^j_{\mathfrak{m}_2}(A_2)_\beta \end{aligned}$$

as required. □

We will denote the a-invariant of a graded k-algebra R by $a(R) = \max\{j : H^{\dim R}_{\mathfrak{m}}(R)_j \neq 0\}$.

Notice that, since $A \cong G$, Cohen-Macaulayness of the Rees algebra is trivial by Theorem 5.1.22 [5]. Namely, R is Cohen-Macaulay if and only if A is Cohen-Macaulay and $a(A) < 0$.

Recall that a finitely generated graded k-algebra R is a generalized Cohen-Macaulay ring if $\ell(H^i_{R_+}(R)) < \infty$ for all $i < \dim R$, where $\ell(-)$ denotes the length. Now we consider generalized Cohen-Macaulayness of the Rees algebra under the condition that A is generalized Cohen-Macaulay. If $I = \mathfrak{m}$, this is immediate from Theorem 4. Namely, if A is generalized Cohen-Macaulay, so is R. We now consider the general case. Before that we prepare a lemma.

Lemma 7. *Let R_1 and R_2 be finitely generated graded k-algebra. Then if $R_1 \otimes_k R_2$ is generalized Cohen-Macaulay ring then so are R_i $(i = 1, 2)$.*

Proof. Let $B := R_1 \otimes_k R_2$, $\mathfrak{m} = R_+$, $\mathfrak{m}_i := (R_i)_+$ $(i = 1, 2)$ and $d := \dim B$. Then we have by Theorem 5

$$0 = H^i_{\mathfrak{m}}(B)_P = \bigoplus_{n_1+n_2=i} (H^{n_1}_{\mathfrak{m}_1}(R_1) \otimes_k H^{n_2}_{\mathfrak{m}_2}(R_2))_P$$

for arbitrary $i \neq d$ and $P \in \operatorname{Spec} B \backslash \{\mathfrak{m}\}$. Since $P = P_1 \otimes_k R_2 + R_1 \otimes_k P_2$ for some primes P_1 and P_2 such that $(P_1, P_2) \neq (\mathfrak{m}_1, \mathfrak{m}_2)$, we have $(H^{n_1}_{\mathfrak{m}_1}(R_1) \otimes_k H^{n_2}_{\mathfrak{m}_2}(R_2))_P \cong H^{n_1}_{\mathfrak{m}_1}(R_1)_{P_1} \otimes_k H^{n_2}_{\mathfrak{m}_2}(R_2)_{P_2}$. Thus we have

$$H^{n_1}_{\mathfrak{m}_1}(R_1)_{P_1} \otimes_k H^{n_2}_{\mathfrak{m}_2}(R_2)_{P_2} = 0$$

for aribtrary $(P_1, P_2) \neq (\mathfrak{m}_1, \mathfrak{m}_2)$ and $n_1 + n_2 \neq d$. From this we know that $H^{n_i}_{\mathfrak{m}_i}(R_i)_{P_i} = 0$ for arbitrary $n_i \neq \dim R_i$ and prime ideal $P_i \neq \mathfrak{m}_i$ $(i = 1, 2)$. $\square$

Now we give a condition for R such that $A \cong G$ to be generalized Cohen-Macaulay.

Theorem 8. *Let $A = k[X_1, \ldots, X_n]/J$ be a graded generalized Cohen-Macaulay ring and $I \subset A$ be such that $A \cong G$. Also let $P_1, \ldots, P_u$ be the minimal primes of A such that $\dim A = \dim A/P_i$. Then the Rees ring $R = A[It]$ is generalized Cohen-Macaulay if and only if one of the following holds:*

(1) $I \subset \bigcap_{i=1}^u P_i$
(2) $I \not\subset \bigcap_{i=1}^u P_i$ *and* $\dim A_2 = 0$
(3) $I \not\subset \bigcap_{i=1}^u P_i$, $\dim A_1 > 0$, $\dim A_2 > 0$, $a(A_1) < 0$, $H^{d_1-1}_{\mathfrak{m}_1}(A_1)_j = 0$ *for all* $j \leq -2$, *and* $H^{d_2-1}_{\mathfrak{m}_2}(A_2) = 0$.

Proof. We have

$$\dim R = \begin{cases} \dim A & \text{if } I \subset \bigcap_{i=1}^u P_i \\ \dim A + 1 & \text{otherwise} \end{cases}$$

where $P_1, \ldots, P_u$ are the minimal prime ideals of A such that $d = \dim A/P_i$ (see [4]).

Now from the short exact sequences

$$0 \longrightarrow R_+ \longrightarrow R \longrightarrow A \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow R_+(1) \longrightarrow R \longrightarrow G \longrightarrow 0,$$

we have, by assumption, the exact sequences

$$0 = H^{i-1}_{\mathfrak{m}}(A)_p \longrightarrow H^i_{\mathcal{M}}(R_+)_P \longrightarrow H^i_{\mathcal{M}}(R)_P \longrightarrow H^i_{\mathfrak{m}}(A)_p = 0$$

and

$$0 = H^{i-1}_{\mathcal{M}}(G)_P \longrightarrow H^i_{\mathcal{M}}(R_+)_P(1) \longrightarrow H^i_{\mathcal{M}}(R)_P \longrightarrow H^i_{\mathcal{M}}(G)_P = 0$$

for $i \neq d, d+1$ $(d := \dim A)$ and $P \in \operatorname{Spec} R \backslash \{\mathcal{M}\}$ with $p = P \cap A$. Thus $H^i_{\mathcal{M}}(R)_P(1) \cong H^i_{\mathcal{M}}(R)_P$ for such P and i. Then we have $H^i_{\mathcal{M}}(R)_P = H^i_{\mathcal{M}}(R)_P(1)$ for $i \neq d, d+1$ and $P \neq \mathcal{M}$. Then, since $H^i_{\mathcal{M}}(R)$ is Artinian, we have $H^i_{\mathcal{M}}(R)_P = 0$, i.e., $\ell(H^i_{\mathcal{M}}(R)) < \infty$ for $i = 0, \ldots, d-1$. Thus R is generalized Cohen-Macaulay if $\dim R = \dim A$.

We know consider the case of $\dim R = \dim A + 1$. We set $d_i := \dim A_i$ $(i = 1, 2)$ and we then have $d = d_1 + d_2$. By Proposition 6, we have

$$\begin{aligned}
&H^d_{\mathcal{M}}(R)_a \\
&= \bigoplus_{\substack{\alpha+\beta=a\\ \alpha\geq 0}} H^{d_1}_{\mathfrak{m}_1}(A_1)^{\oplus \alpha+1}_{\alpha} \otimes_k H^{d_2}_{\mathfrak{m}_2}(A_2)_{\beta} \\
&\quad \oplus \bigoplus_{\substack{\alpha+\beta=a\\ \alpha\leq -2}} \left[H^{d_1-1}_{\mathfrak{m}_1}(A_1)^{\oplus -\alpha-1}_{\alpha} \otimes_k H^{d_2}_{\mathfrak{m}_2}(A_2)_{\beta} \oplus H^{d_1}_{\mathfrak{m}_1}(A_1)^{\oplus -\alpha-1}_{\alpha} \otimes_k H^{d_2-1}_{\mathfrak{m}_2}(A_2)_{\beta} \right]
\end{aligned}$$

so that

$$\begin{aligned}
&H^d_{\mathcal{M}}(R) \\
&\quad = \bigoplus_{a\in\mathbb{Z}} H^d_{\mathcal{M}}(R)_a \\
&\quad = \bigoplus_{\alpha\geq 0} H^{d_1}_{\mathfrak{m}_1}(A_1)^{\oplus \alpha+1}_{\alpha} \otimes_k H^{d_2}_{\mathfrak{m}_2}(A_2) \oplus \bigoplus_{\alpha\leq -2} H^{d_1-1}_{\mathfrak{m}_1}(A_1)^{\oplus -\alpha-1}_{\alpha} \otimes_k H^{d_2}_{\mathfrak{m}_2}(A_2) \\
&\qquad \oplus \bigoplus_{\alpha\leq -2} H^{d_1}_{\mathfrak{m}_1}(A_1)^{\oplus -\alpha-1}_{\alpha} \otimes_k H^{d_2-1}_{\mathfrak{m}_2}(A_2).
\end{aligned}$$

Now by Lemma 7, $H^{d_1-1}_{\mathfrak{m}_1}(A_1)$ and $H^{d_2-1}_{\mathfrak{m}_2}(A_2)$ are of finite length. Also $H^{d_1}_{\mathfrak{m}_1}(A_1)$ and $H^{d_2}_{\mathfrak{m}_2}(A_2)$ are non-zero Artinian modules. Thus if $\ell(H^{d_2}_{\mathfrak{m}_2}(A_2)) = \infty$, we know that we have $\ell(H^d_{\mathcal{M}}(R)) < \infty$ provided

(1) $H^{d_2-1}_{\mathfrak{m}_2}(A_2) = 0$ or $\ell(H^{d_1}_{\mathfrak{m}_1}(A_1)) < \infty$,
(2) $a(A_1) < 0$ and
(3) $H^{d_1-1}_{\mathfrak{m}_1}(A_1)_j = 0$ for all $j \leq -2$.

On the other hand, if $\ell(H^{d_2}_{\mathfrak{m}_2}(A_2)) < \infty$, we have $\ell(H^d_{\mathcal{M}}(R)) < \infty$ provided $H^{d_1-1}_{\mathfrak{m}_2}(A_2) = 0$ or $\ell(H^{d_1}_{\mathfrak{m}_1}(A_1)) < \infty$. By Grothendieck's finiteness theorem (see, for example, Theorem 9.5.2 [1]), we have $\ell(H^{d_i}_{\mathfrak{m}_i}(A_i)) < \infty$ if and only if $\dim A_i = 0$ $(i = 1, 2)$. Thus we have the desired result. □

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